LOCAL INTEGRABILITY OF BESSEL FUNCTIONS ON SPLIT GROUPS

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Abstract. In this paper, we prove that the Bessel functions are locally integrable for all connected split reductive linear algebraic groups over a p-adic field F and the Bessel distributions are given by integrals against these Bessel functions, which are previously known only for $GL(2), GL(3)$ proved by Baruch.

1. Introduction

The main purpose of this paper is to prove the local integrability of Bessel functions attached to irreducible generic smooth representations of G, where G is a connected split reductive linear algebraic group over a p-adic field F, and then the Bessel distributions are given by integrations against these Bessel functions.

Let N_0 be the unipotent radical of a Borel subgroup of G, and ψ_0 be a nondegenerate character on N_0. Let π be an irreducible generic representation of G with contragredient $\widetilde{\pi}$. Use π^* and $\widetilde{\pi}^*$ to denote the linear dual of π and $\widetilde{\pi}$ respectively. Let $f \in C_c^\infty(G)$ be a locally constant function with compact support on G, take $l \in \pi^*$, $l' \in \widetilde{\pi}^*$ to be fixed nonzero Whittaker functionals with respect to ψ_0 and ψ_0^{-1}, respectively. Define $\widetilde{\pi}(f)l'$ as

$$\widetilde{\pi}(f)l' = \int_G f(g)\widetilde{\pi}(g)l'dg$$

or equivalently, for any $\tilde{v} \in \widetilde{\pi}$,

$$\begin{aligned} < \widetilde{\pi}(f)l', \tilde{v} >&= \int_G f(g) < \widetilde{\pi}(g)l', \tilde{v} > dg \\ &= \int_G f(g) < l', \widetilde{\pi}(g^{-1})(\tilde{v}) > dg \end{aligned}$$

then $\widetilde{\pi}(f)l'$ is a smooth linear functional on $\widetilde{\pi}$, hence can be identified with a vector $v_f \in \pi$.

Definition 1.1. *Define Bessel distribution $B(f)$ as*

$$B(f) = l(v_f)$$

Baruch obtained the first regularity result about $B(f)$. By Theorem 2.3 in [Ba01], when restricted to the open Bruhat cell Ω, this Bessel distribution $B(f)$ is given by a locally constant function j_π°, which is called the Bessel function. We extend this function to the whole group G by putting zero outside Ω. It is natural to ask whether j_π° is locally integrable on G and represents the Bessel distribution $B(f)$? Our main result of this paper is to answer this question affirmatively.

Theorem 1.2. *j_π° is locally integrable on G, and for any $f \in C_c^\infty(G)$, we have*

$$B(f) = \int_G f(g) j_\pi^\circ(g) dg.$$

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The Bessel distributions appear naturally in relative trace formulas and Kuznetsov trace formulas, and are analogous to character distributions. Our result is then an analog of Harish-Chandra's regularity theorem on character distributions ([HC65, HC78]). Previously, such regularity results are only known for $GL_2(F), GL_3(F)$ by Baruch in [Ba97, Ba04]. His method is based on Shalika germs of certain Kloosterman integrals on $GL_n(F)$ developed by Jacquet and Yangbo Ye in [Ja, JY96, JY99]. However, it seems difficult to generalize his method to $GL_n(F)$ as Shalika germs become much more complicated. Moreover, this notion is even not available for groups other than $GL_n(F)$.

Built on earlier work of Baruch, Lapid and Zhengyu Mao([Ba05, LM09, LM13]), our method in this paper is different and seems new for similar problems. The first step is to extend Bessel distribution $B(f)$ continuously to the space of rapidly decreasing functions on G, containing $C_c^\infty(G)$, which is a topological vector space and this gives us some estimate on j_π°. The second is to construct a neighborhood for certain element $g \in \Omega$, on which j_π° is locally constant, and to estimate the volume of this neighborhood in terms of norm of g. Combining these preparations one can show the local integrability of Bessel functions.

It would be interesting to see if the methods in the present paper can be used to deal with spherical distributions for p-adic symmetric spaces appearing in relative trace formulas.

This paper is organized as follows. In section 2, we collect some facts which will be useful later. In section 3, we review basics about Bessel functions and Bessel distributions. In section 4, we define space of rapidly decreasing functions $\mathcal{S}(G)$ of G, and extend continuously $B(f)$ to $\mathcal{S}(G)$ in section 5. In section 6, we construct neighborhoods for elements in Ω and estimate its volume. The main result is finally proved in section 7.

2. Notations and Preliminary

F denotes a p-adic field, with ring of integers $\mathcal{O}$. v_F is the p-adic valuation of F and $|\cdot|$ is the p-adic absolute value on F. Let q be the cardinality of its residue field F_q. Use ϖ to denote a fixed uniformizer of F, and let $\mathfrak{p}$ be the maximal ideal it generates in $\mathcal{O}$. $\psi : F \to \mathbb{C}^*$ is an additive character with conductor exactly $\mathcal{O}$.

Let A be a maximal $F-$split torus of G, B be a Borel subgroup of G containing A. N_0 is the unipotent radical of B. Let $\bar{N}_0, \bar{B}$ be the opposite of N_0, B. Use $\mathfrak{g}, \mathfrak{a}$ to be the Lie algebras of G, A respectively. We denote by Φ the system of roots of the pair (G, A), by Φ^+ the set of positive roots and Δ the set of simple roots. For each root $\alpha \in \Phi$, define $\mathfrak{g}^\alpha$ to be

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X, \quad \forall H \in \mathfrak{a}\}.$$

Choose $X_\alpha \in \mathfrak{g}^\alpha$ to be a base of $\mathfrak{g}^\alpha$ defined over F, such that the system $\{X_\alpha\}$ is a Chevalley base of $\mathfrak{g}$. We define in the same way the subgroup N_0^α of G and homomorphism $x_\alpha : F \to N_0^\alpha$ with

$$hx_\alpha(t)h^{-1} = x_\alpha(\alpha(h)t)$$

for all $t \in F, h \in A$. Each $u \in N_0$ can be written as

$$u = \prod_{\alpha \in \Phi^+} x_\alpha(t_\alpha).$$

For roots α, β with $\alpha + \beta \neq 0$, we have the commutator relation

$$(x_\alpha(t), x_\beta(s)) = \prod_{i>1, j>1} x_{i\alpha + j\beta}(c_{i,j} t^i s^j). \tag{2.1}$$

It follows that if α is simple then t_α doesn't depend on the order in the above product of u, and thus we can define

$$\psi_0(u) = \prod_{\alpha \in \Delta} \psi(t_\alpha)$$

which is a nondegenerate character of N_0.

For each root α, define a map $\phi_\alpha : N_0^\alpha \to \mathbb{Z} \cup \infty$ by the relation

$$\phi_\alpha \circ x_\alpha = v_F.$$

Define $\phi_0 : A \to \mathbb{Z} \cup \infty$ by setting

$$\phi_0(a) = \inf_{\chi \in X^*(A)} v_F(\chi(h) - 1).$$

where $X^*(A) = Hom(A, F^\times)$ is the lattice of $F-$rational characters. If m is a positive integer, define

$$N_m^\alpha = \phi_0^{-1}([m, \infty]) = x_\alpha(\mathfrak{p}^m)$$
$$A_m = \phi_0^{-1}([m, \infty]) = \{a \in A : \chi(a) \in 1 + \mathfrak{p}^m, \quad \forall \ \ \chi \in X^*(A)\}.$$

Let K_m be the subgroup of G generated by subgroups H_m, N_m^α for all $\alpha \in \Phi$. Put $N_m = N_0 \cap K_m, \bar{N}_m = \bar{N}_m \cap K_m$. We recall the following result of F.Rodier (Lemma 1 in [Ro75] or Lemma 4 in section 2.2.5 of [Ge] for Chevalley groups).

Lemma 2.1. *(1).If $m \geq 1$, then $K_m = \bar{N}_m A_m N_m = N_m A_m \bar{N}_m$.*
(2) $N_m = \prod_{\alpha \in \Phi^+} N_m^\alpha$.
(3) If $m \geq 1$, then K_{2m}/K_m is commutative.
(4) $\cap_{m \geq 1} K_m = \{I\}$.

We will define a character ψ_m on K_m by

$$\psi_m(\bar{b}u) = \prod_{\alpha \in \Delta} \psi(\varpi^{-2m} t_\alpha)$$

if we write any element of K_m as $\bar{b}u$ with $\bar{b} \in \bar{B}_m = \bar{N}_m A_m, u = \prod_{\alpha \in \Phi^+} x_\alpha(t_\alpha) \in N_m$. To check this is indeed a character, consider the map

$$K_m/K_{2m} \to \mathfrak{g}(\mathfrak{p}^m/\mathfrak{p}^{2m}) : A \to A - 1.$$

By (19) in Lemma 5 of section 2.2.5., Proposition 1 in section 2.2.7. and Lemma 5 in section 2.2.6. of [Ge], we know the above map is an isomorphism. Hence if $h, g \in K_m$, then

$$hg = (h-1) + (g-1) + 1 \pmod{K_{2m}}.$$

Note that in the expression

$$\bar{b}u = \bar{b} \prod_{\alpha \in \Phi^+} x_\alpha(t_\alpha),$$

t_α is independent of the order of this decomposition if α is a simple root again by the commutator relation (2.1). So for α simple we have

$$(hg)_\alpha = h_\alpha + g_\alpha \pmod{\mathfrak{p}^{2m}},$$

and it follows that ψ_m is a well defined character on K_m.

Fix an element $d \in A$ satisfying $\alpha(a) = \varpi^{-2}$ for all $\alpha \in \Delta$. Put $J_m = d^m K_m d^{-m}$ and define a character, still denoted as ψ_m, on J_m by $\psi_m(j_m) = \psi_m(d^{-m}\bar{b}ud^m)$ where $j_m = d^m \bar{b}ud^{-m} \in J_m$. By Lemma 2 in [Ro75], ψ_m is the same as ψ_0 when restricted to $J_m \cap N_0$.

Let K be a hyperspecial maximal compact subgroup of G. We will choose an embedding $\tau : G \to GL_n(F)$ of G into some general linear group $GL_n(F)$ with $\tau(K) \subset GL_n(\mathcal{O})$. Let B_1 be a Borel subgroup of $GL_n(F)$ containing $\tau(B)$. There exists an element $h \in GL_n(\mathcal{O})$ such that hB_1h^{-1} is upper triangular. Consider the composite

$$\tau_0 : G \xrightarrow{\tau} GL_n(F) \xrightarrow{g \to hgh^{-1}} GL_n(F)$$

then $\tau_0(K) \subset GL_n(\mathcal{O})$ and $\tau_0(B)$ is upper triangular. We will identify G with its image under τ_0. For any $g \in G$, write $g = (a_{ij}), g^{-1} = (b_{ij})$, let $||g|| = max_{i,j}\{|a_{ij}|, |b_{ij}|\}$ and $\sigma(g) = log||g||$, then as functions on G, both $||g||, \sigma(g)$ are bi-K-invariant. We also know that $||g|| \geq 1, \sigma(g) \geq 0$ and

$||g|| = ||g^{-1}||, \sigma(g) = \sigma(g^{-1})$.

Denote by P a standard parabolic containing B, N_0, with Levi decomposition $P = MN$. Use Z_M to be the center of M, and δ_P to be modular character of P. The simple roots Δ^P of A on Lie algebra of $M \cap N_0$ is a subset of Δ.

Recall that $X^*(A) = Hom(A, F^\times)$, and $X_*(A) = Hom(F^\times, A)$ is the lattice of cocharacters, which can be identified with $Hom(X^*(A), \mathbb{Z})$. Define the map

$$H : A \to X_*(A)$$

by the relation

$$H(\mu(a)) = v(a)\mu, \mu \in X_*(A), a \in F^\times$$

We need to recall results of Lapid and Mao in [LM09] with some notations. Let V be the space of functions on A and A acts on it by translation. For any character χ of A and $n \in \mathbb{N}$, define

$$\mathcal{F}_{\chi,n} = \{v \in V : (a_1 - \chi(a_1))...(a_n - \chi(a_n)).v = 0 \quad \textit{for all} \;\; a_1, ..., a_n \in A\}$$

and the generalized $\chi-$eigenspace

$$\mathcal{F}_\chi = \bigcup_{n=0}^{\infty} V_{\chi,n}$$

Then $\mathcal{F}_{\chi,n}$ can be identified with $\{\chi(a)Q(H(a))\}$ where Q ranges over the polynomials on $X_*(A)$ of degree $< n$.

Use $\mathbf{A}^1$ to denote the affine line. For any standard parabolic subgroup P put

$$\mathbb{M}_P = \prod_{\alpha \in \Delta} \begin{cases} \mathbf{G}_m & \alpha \in \Delta^P \\ \mathbf{A}^1 & \textit{otherwise} \end{cases}$$

and set $\tau : A \to \mathbb{M}_G \hookrightarrow \mathbb{M}_P$ be the homomorphism by $\tau(a)_\alpha = \alpha(a)$ for $\alpha \in \Delta$. Let $\mathcal{S}(\mathbb{M}_P)$ be the space of Schwartz functions on $\mathbb{M}_P$. For any P and $\chi \in \widehat{Z_M}$, we denote by $\mathfrak{F}_{P,\chi,n}$ the space of smooth functions on A spanned by functions of the form

$$\xi(a)\phi(\tau(a))$$

where $\xi \in \mathcal{F}_{\chi,n}$ and $\phi \in \mathcal{S}(\mathbb{M}_P)$.

If π is a smooth representation of G, let $J_P(\pi)$ be the Jacquet module of π with respect to π, which is a smooth representation of M. View it as a representation of Z_M, and set

$$\mathcal{E}_P(\pi) = \{\chi \in \widehat{Z_M} : J_P(\pi)_\chi \;\; \textit{contains a supercuspidal constituent of } M\}$$

which is called the set of cuspidal exponents of π along P. Put the exponents of π to be

$$\mathcal{E}(\pi) = \bigcup_P \mathcal{E}_P(\pi)$$

which is finite if π is of finite length.

The main result of Lapid and Mao (Theorem 3.1 in [LM09]) about asymptotics of Whittaker functions can be stated as follows.

Proposition 2.2. *Let π be an irreducible generic smooth representation of G, and $\mathcal{W}(\pi, \psi_0)$ be the associated Whittaker model. Then there exists a positive integer n such that any $W \in \mathcal{W}(\pi)$ can be written as*

$$W(uak) = \psi_0(u) \sum_{P \supseteq B} \delta_P^{1/2}(a) \sum_{\chi \in \mathcal{E}_P(\pi)} \phi_{P,\chi}(t, k), \quad \forall u \in N_0, a \in A, k \in K,$$

where $\phi_{P,\chi}(\cdot, k) \in \mathfrak{F}_{P,\chi,n}$ for all $k \in K$ and $\phi_{P,\chi}$ is invariant under an open subgroup of K.

3. Bessel functions and Bessel distributions

Fix a representative ω_0 of the longest Weyl group element in G. Put $\Omega := N_0 A\omega_0 N_0$ to be the open Bruhat cell of G. By Theorem 5.7 and Theorem 7.3 in [Ba05] and Theorem 2 in [LM13], for any $g \in \Omega$, any $W \in \mathcal{W}(\pi, \psi_0)$, there exists an open compact subgroup $U \subset N_0$, such that the function

$$\frac{1}{vol(U)} \int_U W(\omega_0 vu)\psi_0^{-1}(u)du$$

is compactly supported as a function in $v \in N_0$, which implies the integral

$$\int_{N_0}^{*} W(gv)\psi_0^{-1}(v)dv := \int_{N_0} \frac{1}{vol(U)} \int_U W(\omega_0 vu)\psi_0^{-1}(u)du\psi_0^{-1}(v)dv$$

exists. Moreover the map

$$W \to \int_{N_0}^{*} W(gv)\psi_0^{-1}(v)dv$$

is a Whittaker functional on π, and

$$\int_{N_0}^{*} W(gv)\psi_0^{-1}(v)dv = j_\pi(g)W(I)$$

for some scalar $j_\pi(g)$ by the uniqueness of Whittaker functionals. This defines a function $j_\pi(g)$ on Ω.

If $\phi \in C_c^\infty(G)$, define the orbital integral $I(\phi, g)$ for $g \in N_0\omega_0 AN_0$ as

$$I(\phi, g) = \int_{N_0} W_\phi(gu)\psi_0^{-1}(u)du.$$

It is known from [JY96] that this orbital integral converges absolutely and defines a locally constant function in g on the open Bruhat cell $N_0\omega_0 AN_0$. By Theorem 1.7 in [Ba05] and Theorem 4 in [LM13], $j_\pi(g)$ is locally given by $I(\phi, g)$ and hence is also locally constant.

The main result in this section is to prove that $j_\pi(g) = j_\pi^\circ(g)$ for any $g \in \Omega$. Before the proof, we need the following important result of Lapid and Zhengyu Mao (Proposition 2.3 in [LM15]). Let $U_1 \subset U_2 \subset \ldots$ be an increasing filtration of compact open subgroups of N_0.

Proposition 3.1. *Suppose (π, V) is an irreducible generic smooth representation of G with contragredient $(\tilde{\pi}, \widetilde{V})$. Let K_0 be an open compact subgroup of G. If $v \in V, \tilde{v} \in \widetilde{V}$ stabilized by K_0, then the matrix coefficient $(\pi(u)v, \tilde{v})$ is compactly supported as a function $u \in N_0$ after averaging. Moreover, there exists a compact open subgroup U_j of N_0, depending only on K_0, such that the limit*

$$\lim_{i\to\infty} \int_{U_i} (\pi(u)v, \tilde{v})\psi_0^{-1}(u)du$$

stabilizes at U_j. The bilinear form

$$\int_{N_0}^{*} (\pi(u)v, \tilde{v})\psi_0^{-1}(u)du := \lim_{i\to\infty} \int_{U_i} (\pi(u)v, \tilde{v})\psi_0^{-1}(u)du$$

is nonzero and is $(N_0, \psi_0)-$equivariant in v and $(N_0, \psi_0^{-1})-$equivariant in $\tilde{v}$.

Theorem 3.2. *If π is generic, then for any $g \in \Omega$, we have*

$$j_\pi(g) = j_\pi^\circ(g).$$

Proof. For $g = u_1\omega_0 a u_2 \in \Omega, u_1, u_2 \in N_0, a \in A$, choose locally constant functions with compact supports $f_1(u), f_2(u)$ on N_0, f_3 on $\omega_0 A$. Put $f = f_1 \otimes f_2 \otimes f_3$, then $f \in C_c^\infty(\Omega) \subset C_c^\infty(G)$, and

$$\begin{aligned} B(f) &= \int_G f(g) j_\pi^\circ(g) dg \\ &= \int_{N_0} f_1(u)\psi_0(u)du \int_{N_0} f_2(u)\psi_0(u)du \int_A f_3(\omega_0 b) j_\pi^\circ(\omega_0 b)\delta(b)db \\ &= \int_A f_3(\omega_0 b) j_\pi^\circ(\omega_0 b)\delta(b)db \end{aligned}$$

where f_1, f_2 are chosen so that the first two integrals are equal to 1. For f_3, as both $j_\pi^\circ(g)$ and $j_{\tilde{\pi}}(g^{-1})$ are locally constant, we take f_3 as a multiple of characteristic function of a neighborhood of $\omega_0 a$ on which both $j_\pi^\circ(g)$ and $j_{\tilde{\pi}}(g^{-1})$ are locally constant. Moreover we require it satisfying that the last integral is $j_\pi^\circ(\omega_0 a)$, that is,

$$B(f) = j_\pi^\circ(\omega_0 a).$$

On the other hand, by the above Proposition 3.1, choose $\tilde{v}_0 \in \widetilde{V}$ with $W_{\tilde{v}_0}(I) = 1$, we have

$$\begin{aligned} B(f) &= l(v_f) = \int_{N_0}^* (\pi(u)v_f, \tilde{v}_0)\psi_0^{-1}(u)du \\ &= \int_{N_0}^* \int_G f(g) W_{\tilde{v}_0}(g^{-1}u^{-1})dg\psi_0^{-1}(u)du \\ &= \int_{N_0}^* \int_{N_0} f_1(u')\psi_0(u')du' \int_{A\times N_0} f_2(u'') f_3(\omega_0 b) W_{\tilde{v}_0}(b^{-1}\omega_0 u''^{-1}u^{-1})\delta(b)du''db\psi_0^{-1}(u)du. \end{aligned}$$

We can require further that $W_{\tilde{v}_0} \in \Omega^\circ(N_0\backslash G, \psi_0)$, with $\Omega^\circ(N_0\backslash G, \psi_0)$ the same notation as the one defined on page 531 in [LM13]. Since f_3 is supported around $\omega_0 a$, by Proposition 1 and Lemma 1 in section 2.6 of [LM13], $u''^{-1}u^{-1}$ is compactly supported in N_0, and thus u is compactly supported. This shows that

$$(\pi(u)v_f, \tilde{v}_0)\psi_0^{-1}(u) = \int_G f(g) W_{\tilde{v}_0}(g^{-1}u^{-1})dg\psi_0^{-1}(u)$$

is in fact a compactly supported function in u. Then in the integral

$$\int_{N_0} \int_G f(g) W_{\tilde{v}_0}(g^{-1}u^{-1})dg\psi_0^{-1}(u)du$$

both g, u belong to some compact open subsets, and hence the integral is absolutely convergent. This implies that

$$\begin{aligned} B(f) &= \int_{N_0} \int_G f(g) W_{\tilde{v}_0}(g^{-1}u^{-1})dg\psi_0^{-1}(u)du \\ &= \int_G f(g) \int_{N_0} W_{\tilde{v}_0}(g^{-1}u^{-1})\psi_0^{-1}(u)dudg \\ &= \int_G f(g) j_{\tilde{\pi}}(g^{-1})dg \\ &= j_{\tilde{\pi}}(a^{-1}\omega_0). \end{aligned}$$

Hence by Lemma 2.1 in [FLO12] we have

$$j_{\tilde{\pi}}^\circ(a^{-1}\omega_0) = j_\pi^\circ(\omega_0 a) = j_{\tilde{\pi}}(a^{-1}\omega_0)$$

which proves the theorem. $\square$

4. Space of rapidly decreasing functions

In this section, we introduce and study the basic properties of the space of rapidly decreasing functions $\mathcal{S}(G)$ which is the p-adic analogue of space of rapidly decreasing functions on real reductive groups in section 7.1 of [W88].

Let K_0 be an open compact subgroup of G, for a positive real number $r > 0$, if f is a smooth bi-K_0-invariant function on G, define

$$\nu_v(f) = \sup_{g \in G} |f(g)| ||g||^r$$

Consider

$$\mathcal{S}_{K_0} = \{f \quad \textit{is smooth and bi-}K_0\textit{-invariant} : \nu_r(f) < \infty \quad \textit{for any } r > 0.\}$$

Then $\{\nu_r : r > 0\}$ is a family of seminorms on $\mathcal{S}_{K_0}$. The topology of $\mathcal{S}_{K_0}$ is generated by the balls $U_{fr\epsilon}$ of the form

$$U_{fr\epsilon} = \{\phi \in \mathcal{S}_{K_0} : \nu_r(f - \phi) < \epsilon\}$$

and $\mathcal{S}_{K_0}$ becomes a Hausdorff locally convex topological vector space.

If $K_1 \subset K_2$ are open compact subgroups, then $\mathcal{S}_{K_2} \subset \mathcal{S}_{K_1}$.

Lemma 4.1. *$\mathcal{S}_{K_2}$ is closed in $\mathcal{S}_{K_1}$.*

Proof. For $f \in \mathcal{S}_{K_1} \backslash \mathcal{S}_{K_2}$, there exist $g_0 \in G, k_0 \in K_2 \backslash K_1$ with $f(g_0k_0) \neq f(g_0)$ (or $f(k_0g_0) \neq f(g_0)$). Consider the following ball with ϵ to be determined later

$$U_{f1\epsilon} = \{\phi \in \mathcal{S}_{K_1} : \nu_1(f - \phi) < \epsilon\}$$

Thus for $\phi \in U_{f1\epsilon}$, we have

$$|\phi(g) - f(g)| \leq \frac{\epsilon}{||g||} \qquad \forall g \in G$$

Then since

$$|\phi(g_0k_0) - \phi(g_0)| + |\phi(g_0k_0) - f(g_0k_0)| + |f(g_0) - \phi(g_0)| \geq |f(g_0k_0) - f(g_0)|$$

hence

$$\begin{aligned} |\phi(g_0k_0) - \phi(g_0)| &\geq |f(g_0k_0) - f(g_0)| - |\phi(g_0k_0) - f(g_0k_0)| - |f(g_0) - \phi(g_0)| \\ &> |f(g_0k_0) - f(g_0)| - \frac{\epsilon}{||g_0||} - \frac{\epsilon}{||g_0k_0||} \end{aligned}$$

Now choose ϵ small enough so that

$$|f(g_0k_0) - f(g_0)| - \frac{\epsilon}{||g_0||} - \frac{\epsilon}{||g_0k_0||} > 0$$

which implies that $\phi(g_0k_0) \neq \phi(g_0)$, and $U_{f1\epsilon} \subset \mathcal{S}_{K_1} \setminus \mathcal{S}_{K_2}$. Similarly for the case $f(k_0g_0) \neq f(g_0)$. Hence the lemma follows. □

Now choose a system of compact open neighborhoods $\{K_n\}_{n=1}^{\infty}$ of I in G with $K_1 \supset K_2 \supset \cdots$. Then $\mathcal{S}_{K_1} \subset \mathcal{S}_{K_2} \subset \cdots$. Let

$$\mathcal{S}(G) = \bigcup_i \mathcal{S}_{K_i}$$

Then $(\mathcal{S}(G), \{\mathcal{S}_{K_i}\})$ form an inductive system in the sense of section 5.1 in [Con]. Moreover, by the above lemma this is a strict inductive system as in the definition 5.12 in [Con]. We give the inductive limit topology on $\mathcal{S}(G)$, which is exactly the topology defined by the seminorms ν_r on $\mathcal{S}(G)$ in this case as the following lemma claims.

Lemma 4.2. *The inductive limit topology on $\mathcal{S}(G)$ coincides with the topology defined by the seminorms ν_r on $\mathcal{S}(G)$.*

Proof. We will show that $\mathcal{S}(G)$ with the topology determined by the seminorms ν_r satisfy the universal property characterizing the inductive limit topology, see for example Proposition 2.8 in [T].

So let Y be a locally convex topological vector space with a family of continuous linear maps $\psi_i : \mathcal{S}_{K_i} \to Y$ satisfying the following commutative diagrams for any $i > j$

$$\begin{tikzcd} \mathcal{S}_{K_j} \arrow[rr] \arrow[dr, "\psi_j"'] & & \mathcal{S}_{K_i} \arrow[dl, "\psi_i"] \\ & Y & \end{tikzcd}$$

Now define a map $\psi : \mathcal{S}(G) \to Y$. For any $f \in \mathcal{S}(G)$, then $f \in \mathcal{S}_{K_i}$ for some i. Put

$$\psi(f) = \psi_i(f)$$

By the above commutative diagram, ψ is well defined. Moreover ψ is a linear map and it is the unique linear map which makes the following diagrams commutative for all i

$$\begin{tikzcd} \mathcal{S}(G) \arrow[rr, "\psi"] & & Y \\ & \mathcal{S}_{K_i} \arrow[ul] \arrow[ur, "\psi_i"'] & \end{tikzcd}$$

Finally note that the restriction of the topology determined by ν_r to each $\mathcal{S}_{K_i}$ coincides with the topology on $\mathcal{S}_{K_i}$, and the restriction of ψ to each $\mathcal{S}_{K_i}$ is nothing but ψ_i, which is continuous. Thus by Proposition 5.7 in [Con], ψ is continuous. This proves that $\mathcal{S}(G)$ with the seminorm topology has the universal property of inductive limit topology. $\square$

5. Extension of Bessel distributions

In this section, we will show that the Bessel distribution $B = B_\pi$ attached to an irreducible generic representation π of G can be extended to a continuous linear functional on $\mathcal{S}(G)$.

So let (π, V_π) be an irreducible generic admissible smooth representation of G with its contragredient denoted by $(\widetilde{\pi}, V_{\widetilde{\pi}})$. l, l' are fixed nonzero Whittaker functionals on $\pi, \widetilde{\pi}$ with respect to ψ, ψ^{-1} respectively. For $f \in \mathcal{C}_c^\infty(G)$, $W_{\widetilde{v}} \in \mathcal{W}(\widetilde{\pi}, \psi^{-1})$, the integral

$$\int_G f(g) W_{\widetilde{v}}(g^{-1}) dg$$

converges absolutely and defines a smooth linear functional on $\widetilde{\pi}$, hence can be identified with a vector $v_f \in V_\pi$. Then the Bessel distribution $B = B_{l,l'}$ is given by

$$B(f) = B_\pi(f) := l(v_f).$$

Lemma 5.1. *If $r > 0$ is large enough, then for any $W_{\widetilde{v}} \in \mathcal{W}(\pi, \psi^{-1})$, the integral*

$$\int_G \frac{1}{||g||^r} W_{\widetilde{v}}(g^{-1}) dg$$

converges absolutely.

Proof. Since $||g|| = ||g^{-1}||$, take a change of variable $g \to g^{-1}$ and replace r by $6r$, we have

$$
\begin{aligned}
& \int_G \frac{1}{||g||^{6r}} |W_{\widetilde{v}}(g^{-1})| dg \\
= \quad & \int_G \frac{1}{||g||^{6r}} |W_{\widetilde{v}}(g)| dg \\
= \quad & \int_{N_0 \times A \times K} \frac{1}{||uak||^{6r}} |W_{\widetilde{v}}(uak)| \delta_0^{-1}(a) du d^\times a dk \\
\leq \quad & \int_{N_0 \times A \times K} \frac{1}{||ua||^{6r}} |\sum_{P \supseteq B} \delta_P^{1/2}(a) \sum_{\chi \in \mathcal{E}_P(\pi)} \phi_{P,\chi}(t,k)| \delta_0^{-1}(a) du d^\times a dk \qquad (by \;\; Proposition \;\; 2.2) \\
\leq \quad & c_0 \int_{N_0} \frac{1}{||u||^{3r}} du \left(\sum_{P \supseteq B} \sum_{\chi \in \mathcal{E}_P(\pi)} \int_{A \times K} |\delta_P^{1/2}(a) \phi_{P,\chi}(t,k)| \frac{\delta_0^{-1}(a)}{||a||^{3r}} d^\times a dk \right)
\end{aligned}
$$

where c_0 is some positive constant and the last step follows from Lemma II.3.1. in [Wal03] as $\sup\{\sigma(a), \sigma(u)\} \geq (\sigma(a) + \sigma(u))/2$.

Consider first the integral

$$\int_{N_0} \frac{1}{||u||^{3r}} du.$$

If $||u|| = q^l, l \geq 0$, then the measure

$$meas\{u \in N_0 : ||u|| = q^l\}$$

is bounded by $C_0 q^l$ where C_0 is a constant depending only on the group. Hence if r is large, we have

$$
\begin{aligned}
& \int_{N_0} \frac{1}{||u||^{3r}} du \\
\leq \quad & C_0 \sum_l \frac{1}{q^{3lr-l}} < \infty.
\end{aligned}
$$

Now to consider the rest terms, as the sum in the bracket has only finitely many terms, it suffices to consider a single term. Since $\phi_{P,\chi}(a,k) = \chi(a) Q(H(a)) \varphi(\tau(a), k)$ with Q is a polynomial on $X_*(A)$ with degree $< n$, and $\varphi(\cdot, k)$ is a Schwartz function on $\mathcal{S}(\mathbb{M}_P)$ for each $k \in K$ and invariant under an open subgroup of K. Then

$$
\begin{aligned}
& \int_{A \times K} |\delta_P^{1/2}(a) \phi_{P,\chi}(t,k)| \frac{\delta_0^{-1}(a)}{||a||^{3r}} d^\times a dk \\
= \quad & \int_{A \times K} |\chi_0(a)||Q(H(a))||\varphi(\tau(a),k)| \frac{1}{||a||^{3r}} d^\times a dk \qquad\qquad (\chi_0 = \delta_P^{1/2} \delta_0^{-1} \chi).
\end{aligned}
$$

Note that the kernel of τ is Z_G. Let $A' = A/Z_G$. Since $\varphi(\tau(a),k)$ is invariant under some open compact subgroup of K, so the integration with respect to k is in fact a finite sum, and it suffices to consider a single term, with which the integral can be rewritten as

$$
\begin{aligned}
& \int_{A' \times Z_G} |\chi_0(a'z)||Q(H(a'z))||\varphi(\tau(a'z))| \frac{1}{||a'z||^{3r}} d^\times a' d^\times z \\
= & \int_{A' \times Z_G} |\chi_0(a'z)||Q(H(a'z))||\varphi(\tau(a'))| \frac{1}{||a'z||^{3r}} d^\times a' d^\times z.
\end{aligned}
$$

As $\varphi(\tau(a'))$ is a Schwartz function on $\mathcal{S}(\mathbb{M}_P)$, which implies that it is bounded by some positive constant C, and the integral is then dominated by

$$
\begin{aligned}
\leq \quad & C \int_{A' \times Z_G} |\chi_0(a'z)||Q(H(a'z))| \frac{1}{||a'z||^{3r}} d^\times a' d^\times z \\
= \quad & C \int_A |\chi_0(a)||Q(H(a))| \frac{1}{||a||^{3r}} d^\times a
\end{aligned}
$$

As Q has bounded degree less than n, there exists a positive integer r_0, depending only on the representation π, such that if $r \geq r_0$ then the above integral is bounded from the above by

$$\int_A \frac{1}{||a||^r} d^\times a.$$

Now by Lemma 4.1.2. in [S] as $\sigma(a) \leq ||a||$, or use the method there, this integral is finite whenever r is large enough, which finishes the proof.

□

Now choose $f \in \mathcal{S}(G)$, for any $\widetilde{v} \in \widetilde{\pi}$, the integral

$$\int_G f(g) W_{\widetilde{v}}(g^{-1}) dg$$

converges absolutely by the above lemma, which defines a linear functional on $\widetilde{\pi}$. As f is bi-invariant under some open compact subgroup K_0 of G, this linear functional is smooth, and thus can be identified with $v_f \in V_\pi$. We then extend the definition of B to $\mathcal{S}(G)$,

Definition 5.2. *For $f \in \mathcal{S}(G)$, define $B(f) = l(v_f)$.*

Corollary 5.3. *If $G_n = GL_n(F)$, N_n the standard maximal unipotent subgroup of G_n, $f \in \mathcal{S}(G_n)$, $\widetilde{v} \in V_{\widetilde{\pi}}$. Then*

$$\int_{G_n} f(g) W_{\widetilde{v}}(g^{-1}) dg = \int_{N_{n-1}\backslash G_{n-1}} B\left(L\begin{pmatrix} h & \\ & 1\end{pmatrix}.f\right) W_{\widetilde{v}}\begin{pmatrix} h & \\ & 1\end{pmatrix} dh$$

where L denotes the left action of G_n on f.

Proof. By Lemma 5.1, the corollary then follows in the same way as Lemma 3.2 in [Ch15]. □

Proposition 5.4. *The above defined Bessel distribution B is continuous on $\mathcal{S}(G)$.*

Proof. By Proposition 5.7 in [Con], it suffices to show that $\mathcal{S}_{K_i} \hookrightarrow \mathcal{S}(G) \xrightarrow{B} \mathbb{C}$, which is exactly the restriction of B to $\mathcal{S}_{K_i}$, is continuous. Thus if a net $f_\alpha \to f$ in $\mathcal{S}_{K_i}$ with α in some index set, then $\nu_r(f_\alpha - f) \to 0$ for every $r > 0$, and we need to show that $B(f_\alpha) \to B(f)$, that is, $l(v_{v_\alpha}) \to l(v_f)$.

By Proposition 3.1, for some fixed $\tilde{v}_0 \in \widetilde{V}$, the stable limit

$$\lim_{i\to\infty} \int_{U_i} (\pi(u)v, \tilde{v}_0) \psi_{N_0}^{-1}(u) du$$

gives the Whittaker functional l on π. Thus

$$B(f) = \lim_{i\to\infty} \int_{U_i} (\pi(u)v_f, \tilde{v}_0) \psi_0^{-1}(u) du$$

and similarly for $B(f_\alpha)$. As all f, f_α are K_i-invariant, by Proposition 3.1, there exists an open compact subgroup U_j, depending only on K_i, such that $B(f), B(f_\alpha)$ stabilize at U_j, that is,

$$B(f) = \int_{U_j} (\pi(u)v_f, \tilde{v}_0) \psi_0^{-1}(u) du$$

and similarly for $B(f_\alpha)$.

Then

$$\begin{aligned}
& |B(f_\alpha - f)| \\
= \ & |\int_{U_j} ((\pi(u)v_f, \tilde{v}_0) - (\pi(u)v_{f_\alpha}, \tilde{v}_0))\psi_0^{-1}(u)du| \\
= \ & |\int_{U_j}\int_G (f(u^{-1}g) - f_\alpha(u^{-1}g))W_{\tilde{v}_0}(g^{-1})dg\psi_0^{-1}(u)du| \\
\leq \ & \int_{U_j}\int_G |f(u^{-1}g) - f_\alpha(u^{-1}g)||W_{\tilde{v}_0}(g^{-1})|dgdu \\
\leq \ & v_r(f_\alpha - f)\int_{U_j}\int_G \frac{1}{||u^{-1}g||^r}|W_{\tilde{v}_0}(g^{-1})|dgdu.
\end{aligned}$$

As U_j is compact and independent of f, f_α, choose r as large as in Lemma 5.1, then the integral in the above is finite, thus $B(f_\alpha)$ converges to $B(f)$ and B is continuous on $\mathcal{S}(G)$.

□

When restricted to the open Bruhat cell Ω, the Bessel distribution is given by$j_\pi(g) = j_\pi^\circ(g)$ by Theorem 3.2. Assume U_g is a small compact open neighborhood of the identity so that j_π is locally constant on gU_g. As the Bessel distribution $B(f)$ is continuous on $\mathcal{S}(G)$, there exists some $r > 0$, some constant c_r, such that

$$|B(f)| \leq c_r\nu_r(f)$$

for all $f \in \mathcal{S}(G)$. In particular with $f = 1_{gU_g}$ the characteristic function of gU_g, we have

$$|\int_G j_\pi(h)1_{gU_g}(h)dh| \leq c_r\nu_r(1_{gU_g}).$$

Note that $\nu_r(1_{gU_g}) = \sup_{x\in G} 1_{gU_g}(x)||x||^r = ||g||^r$. Then

$$vol(gU_g)|j_\pi(g)| \leq c_r||g||^r \tag{5.1}$$

for some $r > 0$.

6. A Volume Estimation

For diagonal matrix $a = diag(a_1, ..., a_n)$, define $M_1(a) = max\{|a_1/a_2|, |a_2/a_3|, ..., |a_{n-1}/a_n|\}$, $M_2(a) = max\{|a_2/a_1|, |a_3/a_2|, ..., |a_n/a_{n-1}|\} = M_1(a^{-1})$, $M(a) = max\{M_1(a), M_2(a)\}$. It follows that $M(a) \geq 1$ for all a.

For $\phi \in C_c^\infty(G)$ define

$$W(g) = W_\phi(g) := \int_{N_0\times Z_G} \phi(uzg)\psi_0^{-1}(u)\omega_\pi^{-1}(z)dud^\times z.$$

Choose ϕ such that $W(I) = W_\phi(I) = 1$. The following result is very important to us.

Proposition 6.1. *Let $G = GL_n(F)$ and $\omega_0 = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$ be the longest Weyl group element for $GL_n(F)$. For given $W = W_\phi$, $a \in A$, there exist a positive constant $C > 0$ depending only on W and G, and a positive integer $l > 0$ depending only on the group, such that, if $W(\omega_0 au) \neq 0$, then $||u|| \leq C||a||^l$. Moreover, the conclusion still holds if ω_0 is replaced by any other Weyl group element ω.*

Proof. We will first consider ω_0. As W is compactly supported modulo N_0Z_G, there exists finitely many $a_i \in A$, such that
$$supp(W) \subset \cup_i N_0 Z_G a_i K.$$
Without loss of generality, we will assume $supp(W) \subset N_0Z_GK$, from which the general case follows easily.

Hence if $\omega_0 au \in supp(W)$, then there exist $u_1 \in N_0, z \in Z_G, k \in K$ with $\omega_0 au = u_1zk$, and we write $k = u_1^{-1}\omega_0 bu$ with $b = z^{-1}a$. Moreover, by the uniqueness of Bruhat decomposition, u_1, u, b are uniquely determined upon k.

Write $k = (k_{ij}), u = (u_{ij}), b = diag(b_1, b_2, ..., b_n)$. From the relation $k = u_1^{-1}\omega_n bu$ we can express u_{ij} in terms of k_{ij} and b_i. It is a direct computation that we can find $u_{ij}(i < j)$ has the following form.
$$u_{ij} = \sum_{i_1,...,i_r} \frac{k_{i_1j_1}...k_{i_rj_r}}{b_{i_1}...b_{i_r}}.$$
In the above expression, the possible maximal number of summands are bounded by a positive integer L_1, depending only on the group G, and the possible largest value for r is $n-1$.

Now note that $|\frac{b_{i-1}}{b_i}| \leq M(b), i = 2, ..., n$, we get $|b_i| \leq M(b)^{n-i}|b_n|$. By considering the determinant, we have $|b_1...b_n| = 1$, we then have $1/|b_n| \leq M(b)^{n(n-1)/2}$. Then each term in the expression of u_{ij} satisfies
$$|\frac{k_{i_1j_1}...k_{i_rj_r}}{b_{i_1}...b_{i_r}}| \leq \frac{|b_n|^r}{|b_{i_1}...b_{i_r}|}\frac{1}{|b_n|^r}.$$
Now each $\frac{|b_n|}{|b_{i_j}|}$ is bounded by a power of $M(b)$, hence there exists a positive integer $l_{ij} > 0$, such that
$$|\frac{k_{i_1j_1}...k_{i_rj_r}}{b_{i_1}...b_{i_r}}| \leq M(b)^{l_{ij}}.$$
Finally, note that $zb = a$, thus $M(a) = M(b)$. Hence we can find a positive constant $C > 0$ depending only W and the group G, and positive integer $l = max\{l_{ij}\}$ depending only on the group G, such that if $\omega_n au$ is in the support of W, then
$$max\{|u_{ij}| : u = (u_{ij})\} \leq CM(a)^l \leq C||a||^{2l}$$
which proves the result in the case $GL_n(F)$.

When ω_0 is replaced by other Weyl group element, the above arguments applies and one gets the same conclusion. □

Corollary 6.2. *Let G be a connected split reductive linear algebraic group over F. For given $W = W_\phi$, $a \in A$, there exist positive constant $C > 0$ depending only on W and G, and a positive integer $l > 0$ depending only on the group, such that, if $W(\omega_0 au) \neq 0$, then $||u|| \leq C||a||^l$.*

Proof. Recall that G has been identified with a closed subgroup of some general linear group $GL_n(F)$ over F, with $K \subset GL_n(\mathcal{O})$ and B upper triangular. Write the Bruhat decomposition of ω_0 in this $GL_n(F)$ as $\omega_0 = v_1 b\omega v_2$ with some permutation matrix ω in the Weyl group of $GL_n(F)$. Then
$$W(\omega_0 au) = W(v_1 b\omega v_2 au) = \psi_0(v_1)W(b\omega aa^{-1}v_2 au).$$
Note that b, v_2 are uniquely determined by ω_0. Apply the same proof as in the above proposition (not the conclusion of the proposition), the corollary follows. □

• Theorem 5.7 in [Ba05] and Theorem 1 in [LM13] show that, as a function of u, $W(\omega_0 au)$ is compactly supported with support depending only on $M_1(a)$ and W, but without explicit bounds on support.

Recall that if m is a positive integer, $A_m = A \cap K_m = A \cap J_m$. We also set $U_m = N_0 \cap J_m$, $\bar{U}_m = \bar{N}_0 \cap J_m$.

Lemma 6.3. *Choose positive integer m_1 satisfying*
(1) $C||a||^l \leq q^{m_1}$;
(2) $R(A_{m_1}).\phi = \phi$, and thus $R(A_{m_1}).W = W$, where R denotes the right multiplication.
Then $I(\phi, \omega_0 a) = I(\phi, \omega_0 ab)$.

Proof. For any $u \in N_0$, write u uniquely as

$$u = \prod_{\alpha \in \Phi^+} x_\alpha(t_\alpha).$$

For any $b \in A_{m_1}$ and $\alpha \in \Phi^+$, we have

$$b x_\alpha(t_\alpha) b^{-1} = x_\alpha(\alpha(b) t_\alpha).$$

Note that $\alpha(b) \in 1 + \mathfrak{p}^{m_1}$, hence by Proposition 6.1,

$$\psi_0(u) = \prod_{\alpha \in \Delta} \psi(t_\alpha) = \prod_{\alpha \in \Delta} \psi(\alpha(b) t_\alpha) = \psi_0(bub^{-1}).$$

Thus we have

$$\begin{aligned} I(\phi, \omega_0 a) &= \int_{N_0} W(\omega_0 a u) \psi_0^{-1}(u) du \\ &= \int_{N_0} W(\omega_0 a u b) \psi_0^{-1}(u) du \\ &= \int_{N_0} W(\omega_0 a b b^{-1} u b) \psi_0^{-1}(u) du \\ &= \int_{N_0} W(\omega_0 a b b^{-1} u b) \psi_0^{-1}(b^{-1} u b) du \\ &= \int_{N_0} W(\omega_0 a b u') \psi_0^{-1}(u') du' \\ &= I(\phi, \omega_0 ab). \end{aligned}$$

□

Lemma 6.4. *For $g = u_1 \omega_0 a u_2$, choose positive integer m_2 so that*
(3) $q^{m_2} \geq (M(a))^n$;
(4) $u_2 \in U_{m_2}$.
Then for any $\bar{v} \in \bar{U}_{m_2}$, there exist $\bar{v}' \in \bar{U}_{m_2}, u_2' \in U_{m_2}, b' \in A_{m_2}$, such that $u_2 \bar{v} = \bar{v}' b' u_2'$. Moreover $\psi_0(\omega_0 a \bar{v}' a^{-1} \omega_0^{-1}) = 1$, and $\psi_0(u_2) = \psi_0(u_2')$.

Proof. Write $\omega_0 \bar{v}' \omega_0^{-1} = \prod_{\alpha \in \Phi^+} x_\alpha(t_\alpha)$ with $t_\alpha \in \mathfrak{p}^{3m_2}$ if $\alpha \in \Delta$. Then by (3),

$$\omega_0 a \bar{v}' a^{-1} \omega_0^{-1} = \omega_0 a \omega_0^{-1} \omega_0 \bar{v}' \omega_0^{-1} \omega_0 a^{-1} \omega_0^{-1} \in \mathcal{O}$$

and hence $\psi_0(\omega_0 a \bar{v}' a^{-1} \omega_0^{-1}) = 1$.

$\psi_0(u_2) = \psi_0(u_2')$ follows from the fact that ψ_m is a well-defined character on K_m as checked in section 2.

□

As ϕ and W are fixed, there exists a constant M_0 such that if $q^m > M_0$, then $R(A_m).\phi = \phi, R(A_m).W = W$. We will consider those $g = u_1 \omega_0 a u_2$ with $M(a) > M_0$. Now choose positive integer m satisfying

$$C||a||^l \leq q^m, \quad (M(a))^n \leq q^m \quad \text{and} \quad u_2 \in N_m \qquad (**)$$

that is, m satisfies all (1), (2), (3) and (4). We also require m to be the smallest one satisfying (**), i.e., either $C||a||^l > q^{m-1}$, or $M(a) > q^{(m-1)/n}$, or $u_2 \notin U_{m-1}$.

Put $U_g = \bar{U}_m A_m N_K$, where $N_K = N_0 \cap K$, which is an open compact subset of G.

Lemma 6.5. *For any $h \in U_g$, we have $gh \in N_0\omega_0 A N_0$, and $I(\phi, g) = I(\phi, gh)$, where $g = u_1\omega_0 a u_2$.*

Proof. Write $h = \bar{v}bv \in U_g$. Then

$$\begin{aligned}
I(\phi, gh) &= I(\phi, u_1\omega_0 a u_2 \bar{v} b v) \\
&= \psi_0(u_1) I(\phi, \omega_0 a \bar{v}' b' u_2' b v) && \textit{(by Lemma 6.4)} \\
&= \psi_0(u_1)\psi(\omega_0 a \bar{v}' a^{-1}\omega_0) I(\phi, \omega_0 a b' b)\psi_0(b^{-1}u_2' b)\psi(v) \\
&= \psi_0(u_1) I(\phi, \omega_0 a)\psi_0(u_2') && \textit{(by Lemma 6.3 and Lemma 6.4)} \\
&= \psi_0(u_1) I(\phi, \omega_0 a)\psi_0(u_2) = I(\phi, g) && \textit{(by Lemma 6.4).}
\end{aligned}$$

□

Lemma 6.6. *There exists two positive integers b_1, b_2 depending only on the group G, such that $vol(U_g) = vol(\mathcal{O})^{b_1}\dfrac{1}{q^{b_2 m}}$.*

This is an easy calculation and details will be omitted.

Corollary 6.7. *With M_0, then there exist a positive constant C' depending on the group G and W, and a positive constant l' depending only on group G, such that for any $g \in N_0\omega_0 A N_0$ with $M(a) > M_0$, we have*

$$\frac{1}{vol(U_g)} < C'||g||^{l'}.$$

Proof. We know either $C||a||^l > q^{m-1}$, or $M(a) > q^{(m-1)/n}$, or $u_2 \notin U_{m-1}$. In the first case,

$$vol(U_g) = vol(\mathcal{O})^{b_1}\frac{1}{q^{b_2 m}} > vol(\mathcal{O})^{b_1}(Cq)^{-b_2}||a||^{-b_2 l}.$$

The second case is similar to the first case. Now consider the third case, $u_2 \notin U_{m-1}$, which means $u_{ij} \notin \mathfrak{p}^{-b_3 - b_4 m}$ for some positive integers b_3, b_4 depending only on the group G, where $u_2 = (u_{ij})$. This then implies that

$$||u_2|| \geq |u_{ij}| > q^{b_3 + b_4 m}$$

and thus

$$q^m < q^{-\frac{b_3}{b_4}}||u_2||^{\frac{1}{b_4}}.$$

So we have

$$\frac{1}{vol(U_g)} < vol(\mathcal{O})^{-b_1} q^{-\frac{b_2 b_3}{b_4}}||u_2||^{\frac{b_2}{b_4}}.$$

Thus we always have

$$\frac{1}{vol(U_g)} < C_1 max\{||a||, ||u_2||\}^{l_1}$$

with C_1 depending on G and W and l_1 depending only on the group G. Combining with Lemma II.3.1, [Wal03], we get

$$\frac{1}{vol(U_g)} < C_2||au_2||^{l_2}.$$

By Proposition 18.1 in [Ko05], or use the same argument as in Lemma 4.1 of [Zh15], we have

$$\sigma(au_2) \leq \sigma(u_1) + \sigma(au_2) \leq C_3\sigma(u_1\omega_0 a u_2)$$

for all $u_1, u_2 \in N_0, a \in A$ and C_3 is a certain constant depending only on the group. This then gives that

$$\frac{1}{vol(U_g)} < C_4 ||u_1\omega_0 a u_2||^{l_3}$$

with C_4 depending on G and W and l_3 depending only on the group G. $\square$

7. Local Integrability of j_π

In the last section, for g in the open Bruhat cell, we constructed a neighborhood U_g of I such that the orbital integral $I(\phi, h)$ is constant on gU_g, and estimated the volume of U_g in terms of norm of g. To deal with Bessel functions of general generic representations other than supercuspidals, we need to consider similar problems for Bessel functions when restricted to a compact open subset.

Let Y be a compact open subset of G. First by Lemma 8.2 in [Ba05] and Proposition 1 in [LM13], there exists a Whittaker function $W_1 \in \mathcal{W}(\pi, \psi_0)$ such that $W_1(I) = 1$ and

$$j_\pi(g) = \int_{N_0} W_1(gu)\psi_0^{-1}(u)du$$

for all $g \in \Omega = N_0\omega_0 A N_0$. Then by Corollary 9.10 in [Ba05], Theorem 4 in [LM13] and their proofs, we can find some $\phi \in C_c^\infty(G)$, such that $W(1) = W_\phi(1) = 1$ and $W(gu) = W_1(gu)$ for all $g \in Y \cap \Omega$ and $u \in N_0$. This implies that if $g \in Y \cap \Omega$ we have

$$j_\pi(g) = \int_{N_0} W_1(gu)\psi_0^{-1}(u)du = \int_{N_0} W(gu)\psi_0^{-1}(u)du = I(\phi, g).$$

Now as in the last section, there exists a positive constant M_0 such that if $q^m > M_0$, then $R(A_m).\phi = \phi$, $R(A_m).W = W$ and $R(A_m).Y \subset Y$. The last condition can be achieved because Y is a compact open subset. We consider $g = u_1\omega_0 a u_2 \in \Omega \cap Y$ with $M(a) > M_0$. Choose a positive integer m satisfying

$$C||a||^l \leq q^m, \quad (M(a))^n \leq q^m \quad and \quad u_2 \in N_m \qquad (***)$$

then m satisfies the conditions (1),(2),(3) and (4) in Lemma 6.3 and Lemma 6.4 of last section. Put $U_g = \bar{U}_m A_m N_K$. For any $h = \bar{v}bu \in U_g$, write $u_2\bar{v} = \bar{v}'b'u_2'$ as in Lemma 6.4 of last section. Then

$$gh = u_1\omega_0 a u_2 \bar{v} bu = u_1\omega_0 a\bar{v}'b'u_2'bu = u_1\omega_0 a\bar{v}'a^{-1}\omega_0 u_1^{-1} gb'bu_4 = u_3 gb'bu_4$$

for some $u_3, u_4 \in N_0$. By the choice of m, $gb'b \in Y$. Hence

$$W(ghu_5) = W(u_3gb'bu_4u_5) = \psi(u_3)W(gb'bu_4u_5) = \psi_0(u_3)W_1(gb'bu_4u_5) = W_1(ghu_5)$$

for any $u_5 \in N_0$. This then implies that

$$\begin{aligned}
j_\pi(g) &= \int_{N_0} W_1(gu)\psi_0^{-1}(u)du \\
&= \int_{N_0} W(gu)\psi_0^{-1}(u)du \\
&= I(\phi, g) = I(\phi, gh) \\
&= \int_{N_0} W(ghu)\psi_0^{-1}(u)du \\
&= \int_{N_0} W_1(ghu)\psi_0^{-1}(u)du \\
&= j_\pi(gh)
\end{aligned}$$

for all $g \in \Omega \cap Y$ and $h \in U_g$. Moreover we will choose m to be the smallest positive integer satisfying $(***)$, thus Corollary 6.7 is still valid, with the constants C', l' depending not only on W_ϕ, G. We summarize the above in the following proposition.

Proposition 7.1. *Given a compact open subset $Y \subset G$, there exists a positive constant M_0, such that if $g = u_1\omega_0 a u_2$ with $M(a) > M_0$, then there exists a compact open neighborhood U_g of the identity, such that*

$$j_\pi(g) = j_\pi(gh), \quad \forall g \in \Omega \cap Y, \quad h \in U_g$$

and

$$\frac{1}{vol(U_g)} < C'||g||^{l'}$$

where C', l' are constants independent of g.

We are now ready to prove the local integrability of j_π.

proof of Theorem 1.2: For $f \in \mathcal{C}_c^\infty(G)$, suppose $supp(f) \subset Y$ for some compact open subset Y of G. Choose M_0 as in Proposition 7.1, then

$$\begin{aligned} B(f) &= \int_G f(g)j_\pi(g)dg \\ &= \int_\Omega f(g)j_\pi(g)dg \\ &= \int_{M(a)>M_0} f(g)j_\pi(g)dg + \int_{M(a)\le M_0} f(g)j_\pi(g)dg \quad (write \;\; g = u_1\omega_n a u_2 \in \Omega). \end{aligned}$$

For the second integral, write $a = zb$ uniquely with $z \in Z$, b is diagonal with last entry 1, then

$$\begin{aligned} & \int_{M(a)\le M_0} |f(g)j_\pi(g)|dg \\ = & \int_{M(a)\le M_0} |f(u_1\omega_0 zbu_2)\psi_0(u_1)\psi_0(u_2)\omega_\pi(z)j_\pi(\omega_0 b)|\delta(b)du_1du_2d^\times zd^\times b \\ = & \int_{M(b)\le M_0}\int_{N_0\times N_0\times Z} |f(u_1\omega_0 zbu_2)|du_1du_2d^\times z|j_\pi(\omega_0 b)|\delta(b)d^\times b. \end{aligned}$$

As $f \in \mathcal{C}_c^\infty(G)$, the integral

$$I(f,b) := \int_{N_0\times N_0\times Z} |f(u_1\omega_0 zbu_2)|du_1du_2d^\times z$$

converges and defines a locally constant function of b. Also note that $\{b \in A_{n-1} : M(b) \le M_0\}$ is compact, thus the second integral is absolutely bounded by

$$\int_{M(b)\le M_0} I(f,b)|j_\pi(\omega_0 b)|\delta(b)d^\times b$$

and hence is absolutely convergent.

Now look at the first integral. For $g = u_1\omega_0 a u_2$,

$$\begin{aligned} & \int_{M(a)>M_0, g\in\Omega} |f(g)j_\pi(g)|dg \\ = & \int_{M(a)>M_0, g\in\Omega\cap Y} |f(g)j_\pi(g)|dg \\ \le & \; c_r\int_{M(a)>M_0, g\in\Omega\cap Y} |f(g)|||g||^r vol(U_g)^{-1}dg \quad (by \;\; (5.1)) \\ \le & \; c'\int_{M(a)>M_0, g\in\Omega\cap Y} |f(g)|||g||^r||g||^{l'}dg \quad (by \;\; Proposition \;\; 7.1) \\ \le & \; c'\int_G |f(g)|||g||^r||g||^{l'}dg \end{aligned}$$

which is also finite. Thus we get the local integrability of j_π, which proves the first part of Theorem 1.2. The second part is given by the following corollary.

Corollary 7.2. *The Bessel distribution $B(f)$ is given by integration against $j_\pi(g)$, that is, for any $f \in C_c^\infty(G)$,*

$$B(f) = \int_G f(g) j_\pi(g) dg.$$

Proof. For $f \in C_c^\infty(G)$, define distribution

$$B_1(f) = \int_G f(g) j_\pi(g) dg.$$

Because of the local integrability of $j_\pi(g)$, this is well defined. Consider distribution $B(f) - B_1(f)$. By Theorem 3.2, the restriction of this distribution to Ω is zero. Thus it is supported on $G \backslash \Omega$. By Theorem A in [AGS15], the wave front set of distribution $B - B_1$ is contained in $(G \setminus \Omega) \times \mathcal{N}$, where $\mathcal{N}$ is the set of nilpotent elements in the dual of Lie algebra of G. Then by Corollary C in [AGK15], $B - B_1 = 0$, which finishes the proof. ☐

We present two immediate applications of Theorem 1.2.

Corollary 7.3. *Let π_1, π_2 be two generic irreducible smooth representations of G, with associated Bessel functions j_{π_1}, j_{π_2}. If there exists a constant c, such that for all $g \in G$, $j_{\pi_1}(g) = c j_{\pi_2}(g)$, then $\pi_1 \cong \pi_2$.*

Proof. By Lemma 2.2 in [FLO12], the Bessel distributions are linearly independent for two inequivalent irreducible generic representations. The corollary thus follows from Corollary 7.2. ☐

Let G_n be the general linear group $GL_n(F)$, with N_n the standard maximal unipotent subgroup. We embed G_{n-1} into G_n on the left upper corner. Recall that $\omega_0 = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$ is the longest Weyl element of G_n.

Corollary 7.4. *Let π be an irreducible smooth representation of G_n with Bessel function $j_\pi(g)$. If $W_v(g)$ is a Whittaker function in the Whittaker model $\mathcal{W}(\pi, \psi)$ of π, then for any diagonal matrix $h \in G_n$, we have*

$$W_v(y\omega_0) = \int_{N_{n-1} \backslash G_{n-1}} j_\pi \left(y\omega_0 \begin{pmatrix} h^{-1} & \\ & 1 \end{pmatrix} \right) W_v \begin{pmatrix} h & \\ & 1 \end{pmatrix} dh.$$

Proof. Use the same method in Lemma 5.3 of [Ba04], we can show the left side integral is absolutely convergent. The corollary then follows from Theorem 4.2 in [Ch15]. ☐

- The formula in the above corollary essentially gives the action of ω_0 on the Kirillov model of π.

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